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Approximation of Continuous, Periodic Functions by Discrete Linear Positive Operators

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1. INTRODUCTION AND RESULTS

Let Φ_n be a 2π-periodic, continuous, even and non-negative function,[1] satisfying the condition

$$\frac{1}{\pi}\int_0^{2\pi} \Phi_n(t)\,dt = 1$$

for $n = 1, 2,\ldots$. The convolution operators

$$L_n(f, x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\Phi_n(t-x)\,dt, \tag{1.1}$$

where f is a 2π-periodic and continuous function, have been studied extensively and their approximation properties are well known (see [1, Chapter 2; 2, Chapters 1 and 2]). One of the simplest results in that direction can be stated as follows (see [1, p. 32–33; 3]).

THEOREM A. *Let Φ_n be a 2π-periodic, continuous, even and nonnegative function, and let*

$$\tfrac{1}{2} + \sum_{k=1}^{\infty} \rho_{k,n}\cos kx$$

[1] All functions in this paper are real.

be the Fourier series of Φ_n. Then, for every continuous, 2π-periodic f,

$$| L_n(f, x) - f(x)| \leqslant (1 + \pi)\, \omega\left(f, \left(\frac{1 - \rho_{1,n}}{2}\right)^{1/2}\right).$$

Here, $\omega(f, \cdot)$ is the uniform modulus of continuity of f.

If Φ_n is a trigonometric polynomial, then $L_n(f, \cdot)$ is also a trigonometric polynomial whose coefficients can be expressed in terms of the Fourier coefficients of f. Although the degree of approximation by trigonometric polynomials of this type can be very close to the optimal, the effective construction of these polynomials requires numerical evaluation of integrals, which in a certain sense restricts their usefulness from the computational point of view.

Since the evaluation of integrals is based on a quadrature formula, it is clear that most useful approximating polynomials from the computational point of view will be obtained if the integral in (1.1) is replaced by a Riemann sum or by a quadrature formula-type sum, provided this will not change essentially the degree of convergence. The desirability of this approach has been pointed out already in [4].

We shall consider here the operator

$$K_n(f, x) = \frac{2}{m_n + 2} \sum_{k=1}^{m_n+2} f(t_{k,n})\, \Phi_n(t_{k,n} - x), \tag{1.2}$$

where Φ_n is a nonnegative cosine polynomial of degree $\leqslant m_n$ and

$$t_{k,n} = \frac{2k\pi}{m_n + 2}, \qquad k = 1, 2,\ldots, m_n + 2.$$

The operator $K_n(f, \cdot)$ is clearly a Riemann sum approximation of the operator (1.1).

We shall prove here the following result.

THEOREM B. *Let Φ_n be a nonnegative cosine polynomial*

$$\Phi_n(x) = \tfrac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos kx.$$

Then, for every continuous, 2π-periodic f,

$$| K_n(f, x) - f(x)| \leqslant (1 + \pi)\, \omega\left(f\left(\frac{1 - \rho_{1,n}}{2}\right)^{1/2}\right).$$

In actual problems of approximation we shall consider only operators $K_n(f, \cdot)$ whose degree of approximation is close to optimal.

First example of such an operator was given by D. Jackson [5]. Jackson's operator is generated by the kernel

$$\Phi_n(t) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4$$

$$= \frac{1}{2} + \frac{2(n^2-1)}{2n^2+1}\cos t + \sum_{k=2}^{2n-2} \rho_{k,n} \cos kt.$$

Since $m_n = 2n - 2$, we have $t_{k,n} = k\pi/n$, $k = 1, 2,..., 2n$, and a discrete analog of Jackson's operator is the operator

$$K_n(f, x) = \frac{3}{2n^2(2n^2+1)} \sum_{k=1}^{2n} f\left(\frac{k\pi}{n}\right)\left(\frac{\sin(n/2)((k\pi/n) - x)}{\sin \frac{1}{2}((k\pi/n) - x)}\right)^4.$$

Since

$$\rho_{1,n} = \frac{2(n^2-1)}{2n^2+1},$$

from our Theorem B follows that

$$| K_n(f, x) - f(x)| \leqslant (1 + \pi)\, \omega(f, 1/n).$$

Another, still better, example of a discrete operator which gives a degree of convergence close to the optimal, is generated by Korovkin's kernel (see [1, p. 105, 106]):

$$\Phi_n(t) = \frac{1}{n}\sin^2(\pi/n)\left(\frac{\cos(nt/2)}{\cos t - \cos(\pi/n)}\right)^2$$

$$= \tfrac{1}{2} + \cos(\pi/n)\cos t + \sum_{k=1}^{n-2} \rho_{k,n}\cos kt.$$

We have in this case $m_n = n - 2$, $t_{k,n} = 2k\pi/n$, $k = 1,..., n$ and from (1.2) follows that a discrete version of Korovkin's operator is the operator

$$K_n(f, x) = 2\left(\frac{1}{n}\sin\frac{\pi}{n}\right)^2 \sum_{k=1}^{n} f\left(\frac{2k\pi}{n}\right)\left(\frac{\cos(n/2)(2k\pi/n - x)}{\cos(2k\pi/n - x) - \cos \pi/n}\right)^2.$$

Since $\rho_{1,n} = \cos(\pi/n)$, from our Theorem B follows that

$$| K_n(f, x) - f(x)| \leqslant (1 + \pi)\, \omega(f, \pi/2n).$$

2. Proof of Theorem B

Since $K_n(f, \cdot)$ is a linear positive operator on the linear space of 2π-periodic and continuous functions, we have

$$| K_n(f, x) - f(x)| \leqslant (K_n(1, x) + \pi(K_n(1, x))^{1/2})\, \omega(f, \beta_n(x)) \\ + |f(x)|\, | K_n(1, x) - 1 |, \tag{2.1}$$

where $\beta_n^2(x) = K_n(\sin^2((t - x)/2), x)$ (see [1, p. 29, 30]). Thus, we only have to evaluate $K_n(1, x)$ and $\beta_n(x)$. This evaluation is based on the following quadrature formula of the highest possible degree of precision (see [1, p. 20]): For every trigonometric polynomial τ of order $\leqslant n - 1$, we have

$$\frac{2}{n} \sum_{k=1}^{n} \tau\left(\frac{2k\pi}{n}\right) = \frac{1}{\pi} \int_0^{2\pi} \tau(t)\, dt.$$

In particular, if $t_{k,n} = 2k\pi/(m_n + 2)$, $k = 1, 2,..., m_n + 2$, we have, for every trigonometric polynomial τ of order $\leqslant m_n + 1$,

$$\frac{2}{m_n + 2} \sum_{k=1}^{m_n+2} \tau(t_{k,n}) = \frac{1}{\pi} \int_0^{2\pi} \tau(t)\, dt.$$

We shall also use the fact that integrals of a 2π-periodic function over arbitrary intervals of length 2π are all equal.

We shall show first that

$$K_n(1, x) = 1. \tag{2.2}$$

Since $\Phi_n(t - x)$ is a trigonometric polynomial of order m_n, we have

$$\begin{aligned} K_n(1, x) &= \frac{2}{m_n + 2} \sum_{k=1}^{m_n+2} \Phi_n(t_{k,n} - x) \\ &= \frac{1}{\pi} \int_0^{2\pi} \Phi_n(t - x)\, dt \\ &= \frac{1}{\pi} \int_0^{2\pi} \Phi_n(t)\, dt, \end{aligned}$$

and (2.2) follows.

Next,

$$\beta_n^2(x) = \frac{2}{m_n + 2} \sum_{k=1}^{m_n+2} \sin^2 \tfrac{1}{2}(t_{k,n} - x)\, \Phi_n(t_{k,n} - x).$$

Since $\sin^2 \frac{1}{2}(t - x)\, \Phi_n(t - x)$ is a trigonometric polynomial of order $m_n + 1$, we have, by the quadrature formula,

$$\beta_n^2(x) = \frac{1}{\pi} \int_0^{2\pi} \sin^2 \tfrac{1}{2}(t - x)\, \Phi_n(t - x)\, dt$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sin^2 \frac{t}{2}\, \Phi_n(t)\, dt.$$

Hence,

$$\beta_n^2(x) = \tfrac{1}{2}(1 - \rho_{1,n}), \tag{2.3}$$

and Theorem B follows from (2.1), (2.2), and (2.3).

References

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